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(2+1)-AdS Gravity on Riemann Surfaces

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Abstract

We discuss a formalism for solving $(2+1)$ AdS gravity on Riemann surfaces. In the torus case the equations of motion are solved by two functions f and g, solutions of two independent $O(2,1)$ sigma models, which are distinct because their first integrals contain a different time dependent phase factor. We then show that with the gauge choice $k = \sqrt{\Lambda}/tg(2\sqrt{\Lambda}t)$ the same couple of first integrals indeed solves exactly the Einstein equations for every Riemann surface. The $X^A = X^A(x^\mu)$ polydromic mapping which extends the standard immersion of a constant curvature three-dimensional surface in a flat four-dimensional space to the case of external point sources or topology, is calculable with a simple algebraic formula in terms only of the two sigma model solutions f and g. A trivial time translation of this formalism allows us to introduce a new method which is suitable to study the scattering of black holes in $(2+1)$ AdS gravity.

1 Introduction

In this article we extend a previous study of $(2+1)$ gravity on Riemann surface to the case of cosmological constant [1],[2],[3],[4],[5],[6],[7]. The absence of gravitational radiation introduces an important simplification to the dynamics since it allows to choose an instantaneous gauge for the propagation of the gravitational field, and we can forget the problem of the delay due to the speed limitation of signals.

The classical gravitational dynamics reduces to a renormalization of the matter sources, and in particular the two-body interacting problem can be solved as the two particles move following the geodesics around an effective source, whose invariant mass is computable with the Wilson loop of the spin connection around the two particles [7].

The case of the cosmological constant addition is useful not only at a classical level, where it sheds light on black-hole scattering, but also at a quantum level because it teaches us how to quantize the gravitational field when it doesn't carry bulk degrees of freedom [4]. It seems that although non-renormalizable $2+1$ gravity can be understood at the quantum level by virtue of its integrability, of being a topological theory, whose dynamics is to introduce a braiding of a two-dimensional boundary theory, which can be related to a conformal field theory or some integrable deformation of it.

At a quantum level we expect that the gravitational degrees of freedom have lower dimension and live on the boundary of the bulk. Several articles have recently emphasized the importance of the holography in field theory, as the property that allows to reconstruct the field in the bulk starting from the knowledge of the field at the boundary. In $2+1$ gravity the holography property seems particularly easy to show.

Our main aim will be investigate how to reduce classical AdS gravity to a two dimensional field theory. Let us remember rapidly the main results obtained in this direction. At a classical level particle dynamics has already been solved in the gauge $k=0$ [8] since this gauge makes possible to reduce the interacting problem to a choice of conformal mappings, defined by the monodromy conditions, related to the particles' constant of motion. Exact results have been found in the two-body case [8], and an interesting connection with Painlevé VI for the three body case [8, 10]. Furthermore in order to treat topological degrees of freedom we followed Moncrief choice of a time slicing (k constant but not zero)[11]. The resulting equations of motions can be simplified by choosing a conformal gauge for the spatial metric. It has been recognized that the field dynamics for Riemann surfaces can be reduced to a sine-Gordon theory, at least for the conformal factor of the spatial metric, and a first-order formalism based on a first-integral of a $O(2,1)$ sigma model can be built. In this first-order formalism we have built out of the solution f of the $O(2,1)$ sigma model a coordinate transformation $X^a = X^a(x^\mu)$ from a Minkowskian coordinate system to the physical one. Each spatial slice is equivalent to a Riemann surface, which we represent with branch cuts on the complex plane. The corresponding metric is singular at the branch points, which move as particle singularities. The motion of the branch points in the X^A coordinates is free and determined by the Poincaré holonomies, defining the coupling of Riemann surface to gravity. By solving the mapping f, we can find the dynamics of the branch points and of the moduli of the Riemann surface in the physical coordinate system [12].

At a level of the cosmological constant we are going to show that it is possible to avoid further complications, and the first-order formalism can be easily generalized to a couple of first-integrals of two $O(2,1)$ sigma-models, distinct only by a phase factor in the source. As a first consequence we can generalize the classical mathematical theorem regarding the immersion of a Riemann surface with genus $g > 1$ in the Poincare' disk to the immersion of AdS gravity on Riemann surfaces into the direct product of two Poincare' disks. Then the mathematical formalism which has been successfully applied to the $(2+1)$-gravity case can be generalized without problem to the AdS case.

A simple time translation of the gauge choice produces the gauge condition suited to study the scattering of two particles and, a case more interesting, the scattering of black holes. As an outcome of our work we give at disposition an useful reference-point for the study of this non-perturbative classical problem of general relativity, apart from the fact that conformal gauge choice, which is peculiar to all our work, can be done only in the (connected) region outside the horizons.

These are, from our point of view, the more appropriate gauge choices in which to look for a solution for the Riemann surfaces coupled to AdS gravity and the black hole scattering. While for the former ones we give the explicit solution in some cases, for the second problem we postpone to a future work the detailed study.

At a classical level, for example the torus, we recover that its modulus describes a circular motion in Teichmuller space, while for certain Riemann surfaces defined by the condition $f = g$ the solutions to the two sigma models become analytic and the moduli of the Riemann surfaces are static, i.e. they do not have any temporal dependence. Only with a non-analytic solution of the two $O(2,1)$ sigma models it is possible to introduce a non trivial dynamics.

At a quantum level since all these theories have a finite number of degrees of freedom we expect a reduction of quantum field theory to a problem of quantum mechanics. For example we can quantize the Lagrangian of the modulus of the torus, introducing a canonical momentum to τ, and writing down a Schrödinger equation acting on the Hilbert space of square integrable functions of τ [12]. This reduced quantization is analogous to the particle case [4], where one can integrate out the field into an effective action for the particle degrees of freedom. This reduced quantization is an useful short-cut of the full second-quantized problem, i.e. in which one ask himself how to treat topology-changing amplitudes or creation and annihilation processes for particles [1].

Maybe a new treatment of three dimensional gravity resembling two dimensional quantum field theory can handle these problems. In this sense the idea of 't Hooft of quantizing gravity directly in the singular cordinate system seems particularly fruitful, because all the classical results have this common characteristic. Moreover there is in conformal field theory the example of Kniznik with his idea to associate the fields of a conformal field theory to the sheets of a Riemann surfaces. A further investigation in this direction is tempting.

2 York time gauge in the second order formalism

We shall adopt in the following the ADM formalism, since all our result take the assumption that space-time can be globally decomposed as $\Sigma(t) \otimes R$, where $\Sigma(t)$ is a set of space-like surfaces [13, 14]. A particularly meaningful parameterization for the metric revealing the physically relevant degrees of freedom turns out to be this one:

$$ds^2 = \alpha^2 dt^2 - e^{2\phi}|dz - \beta dt|^2, \tag{2.1}$$

where we have chosen conformal coordinates for the spatial metric. This choice of variable, the lapse function α and the shift functions β, is particularly useful when we discuss how to solve the Eulero-Lagrange equations of motion.

The ADM decomposition can be performed at the level of the Einstein-Hilbert action by splitting it into a spatial part, intrinsic to the surfaces $\Sigma(t)$, and an extrinsic part, coming from the embedding , as follows

$$\begin{aligned} S &= -\frac{1}{2}\int \sqrt{|g|}\ R^{(3)}\ d^3x + \Lambda \int \ \sqrt{|g|} d^3x = \\ &- \ \frac{1}{2}\int \sqrt{|g|}\ \left[R^{(2)} + (TrK)^2 - Tr(K^2)\right]\ d^3x + \Lambda \int \ \sqrt{|g|} d^3x, \end{aligned} \tag{2.2}$$

The extrinsic curvature tensor K_{ij} , or second fundamental form of the surface $\Sigma(t)$, is given in terms of the covariant derivatives $\nabla_i^{(2)}$ with respect to the spatial part of the metric:

$$K_{ij} = \frac{1}{2}\sqrt{\frac{|g_{ij}|}{|g|}} \left(\nabla_i^{(2)} g_{0j} + \nabla_j^{(2)} g_{0i} - \partial_0 g_{ij}\right). \tag{2.3}$$

Our aim is to simplify at the maximum level the ADM scheme applied to $(2+1)$ AdS gravity on Riemann surfaces . To achieve such simplification a preliminary step is choosing the conformal coordinates for the spatial metric, which is general if we allow to represent a Riemann surface on a complex plane with branch points (four branch points for a torus, $2g+2$ for an hyperelliptic surface of genus g, and so on). This means that the metric has a singular particle-like behaviour on such branch points, which however must satisfy some integrability condition, such as a finite area $A(t) = \int dz d\overline{z} e^{2\phi}$ on each spatial slice.

The lagrangian for $(2+1)$-gravity, restricted to a spatial metric in conformal gauge, is defined by:

$$\begin{aligned}
\mathcal{L} &= \alpha \nabla^2 \phi + \alpha e^{2\phi} K^2 - \frac{e^{2\phi}}{\alpha} |\partial_z \overline{\beta}|^2 - \Lambda \alpha e^{2\phi} \\
k &= -\frac{1}{2} g^{ij} K_{ij} = \frac{e^{-2\phi}}{2\alpha} [\partial_z (\beta e^{2\phi}) + \partial_{\overline{z}} (\overline{\beta} e^{2\phi}) + \partial_0 e^{2\phi}].
\end{aligned} \tag{2.4}$$

A simplifying feature appears in the equation of motion for β, which reads

$$\partial_{\overline{z}} N + e^{2\phi} \partial_z K = 0, \quad N = \frac{e^{2\phi}}{2\alpha} \partial_z \overline{\beta}. \tag{2.5}$$

From this equation it is clear that the function N is analytic whenever $k = 0$ or $k = k(t)$ is a time-dependent constant. In the following we will show that a convenient choice is

$$k = \frac{\sqrt{\Lambda}}{tg(2\sqrt{\Lambda} t)} \quad \rightarrow \quad \partial_{\overline{z}} N = 0. \tag{2.6}$$

Therefore our gauge choice is defined by the conditions

$$g_{zz} = g_{\overline{z}\overline{z}} = 0 \qquad k = \frac{\sqrt{\Lambda}}{tg(2\sqrt{\Lambda} t)}, \tag{2.7}$$

and thus corresponds to a conformal gauge, with York time $g_{ij} K^{ij} = -2 \frac{\sqrt{\Lambda}}{tg(2\sqrt{\Lambda} t)}$.

The above conditions are enough to eliminate time derivatives from the Lagrangian and to give an instantaneous propagation of the metric, as it appears from the Eulero-Lagrange equations for α and ϕ:

$$\begin{aligned}
\nabla^2 \phi + 4 N \overline{N} e^{-2\phi} &= (k^2(t) + \Lambda) e^{2\phi} \\
\partial_{\overline{z}} N(z) &= 0 \\
\nabla^2 \alpha - 8 N \overline{N} e^{-2\phi} \alpha &= 2(k^2 + \lambda) \alpha e^{2\phi} + 2 \partial_0 k(t) e^{2\phi} = 2(k^2(t) + \Lambda)(\alpha - 2) e^{2\phi}.
\end{aligned} \tag{2.8}$$

To write the last line we have used the property

$$\partial_0 k(t) = -(k^2(t) + \Lambda), \tag{2.9}$$

that works only if

$$k(t) = \frac{\sqrt{\Lambda}}{tg(2\sqrt{\Lambda}t)} \qquad or \qquad k(t) = -\frac{\sqrt{\Lambda}}{cotg(2\sqrt{\Lambda}t)}. \tag{2.10}$$

It turns out hat the first choice is useful to describe the solutions for Riemann surfaces, while the second one is useful to write down solutions for scattering of particles and black holes.

These equations are difficult to solve directly. However, we will show that in the first-order formalism it naturally appears a quite simple structure, related to a couple of $O(2,1)$ σ-models, which automatically solves them.

Although not explicit, the appearance of singularities in the metric produce extra δ function sources, localized on the branch points, in the equations of motion (2.8). The t dependence of the Riemann surface moduli is therefore provided by the covariant conservation of the "underlying" energy-momentum tensor, which in turn implies the geodesic equations for the branch point singularities.

3 Some solutions in the first-order formalism

The first-order formalism gives a direct language for relating holonomies to the physical metric [15, 16]. For example, in [8] we have proposed a non-perturbative solution for the metric and the motion of N interacting spinless particles in $(2+1)$ gravity, based on a harmonic mapping $X^A = X^A(t, z, \overline{z})$ from a regular coordinate system to Minkowskian multivalued coordinates.

Analogously to what we found for the gauge $k = 0$, we are going to solve the gauge condition which corresponds to the solution of the torus coupled to AdS $(2+1)$-gravity , given by

$$k = k(t) = \frac{\sqrt{\Lambda}}{tg(2\sqrt{\Lambda}t)} \qquad g_{zz} = 0. \tag{3.1}$$

The introduction of the cosmological constant makes useless searching for a simplification of the equations of motion in the dreibein formalism [17], while it is fruiful to start from the typical construction of space-times with constant curvature [18], that are obtained as an embedding in a four-dimensional flat metric with signature (++−):

$$ds^2 = dX^A dX^B \eta_{AB} \qquad X^A X_A = \frac{1}{\Lambda}. \tag{3.2}$$

This formalism can also be considered a first-order formalism, defining a new dreibein $E^A_\mu = \partial_\mu X^A$, in which the Lorentz index depends on four coordinates.

An intrinsic local frame of the four-dimensional space-time is given by these four four-vectors $X^A, \partial_z X^A, \partial_{\overline{z}} X^A$ and V^A, which is defined as

$$V^A = 2i\sqrt{\Lambda} e^{2\phi} \epsilon^{ABCD} X_B \partial_z X_C \partial_{\overline{z}} X_D \tag{3.3}$$

orthogonal to the first three ones, and defined to have norm equal to unity. Instead $\partial_z X^A$ and $\partial_{\overline{z}} X^A$ are two null vectors having a non-vanishing scalar product between them. The gauge choice made in the second order formalism corresponds to the following set of equations for the relative immersion X^A:

$$\begin{aligned}
\partial_z \partial_{\overline{z}} X^A &= \frac{\Lambda}{2} e^{2\phi} X^A + \frac{k(t)}{2} e^{2\phi} V^A \\
\partial_z^2 X^A &= 2\partial_z \phi \partial_z X^A + N(z) V^A \\
\partial_z V^A &= 2e^{-2\phi} N(z) \partial_{\overline{z}} X^A + k(t) \partial_z X^A.
\end{aligned} \tag{3.4}$$

We are going to introduce non trivial topology by allowing multivaluedness of X^A. In fact the simplest definition of a genus g Riemann surface coupled to AdS gravity is the quotient of the hyperboloid immersed in the $(2+2)$ flat space-time by a finite set of elements of the $SO(2,2)$ monodromy group, i.e. we identify all the points that can be reached with isometries $(U_i, V_i, i = 1, .., g)$ of the flat metric $ds^2 = dX_0^2 + dX_1^2 - dX_3^2 - dX_4^2$ satisfying

$$\prod_i U_i V_i U_i^{-1} V_i^{-1} = 1. \tag{3.5}$$

This means that circling many times around the cycles $(a_i, b_i, i = 1, ...g)$ of the Riemann surface, the image of a point in the regular coordinate system by the X^A mapping is a lattice of points in the hyperboloid immersed in a flat four-dimensional space-time. The identification of this lattice with a point produces a Riemann surface as spatial slice.

For the torus case, Eq. (3.5) becomes $UV = VU$ and it is solved by an abelian subgroup $Z \otimes Z$ of the $SO(2,2)$ monodromy group, which can be taken as a boost along both the X_0, X_3 direction and in the X_1, X_2 direction. The torus coupled to $(2+1)$ AdS gravity is defined by the following holonomy transformations:

$$U:\begin{cases} X^0 & \rightarrow & ch\lambda_1 X^0 + sh\lambda_1 X^3 \\ X^3 & \rightarrow & ch\lambda_1 X^3 + sh\lambda_1 X^0 \\ X^1 & \rightarrow & ch\lambda_2 X^1 + sh\lambda_2 X^2 \\ X^2 & \rightarrow & ch\lambda_2 X^2 + sh\lambda_2 X^1 \end{cases} \qquad V:\begin{cases} X^0 & \rightarrow & ch\eta_1 X^0 + sh\eta_1 X^3 \\ X^3 & \rightarrow & ch\eta_1 X^3 + sh\eta_1 X^0 \\ X^1 & \rightarrow & ch\eta_2 X^1 + sh\eta_2 X^2 \\ X^2 & \rightarrow & ch\eta_2 X^2 + sh\eta_2 X^1 \end{cases} \tag{3.6}$$

and the flat coordinates satisfy the usual quadratic relation

$$(X^0)^2 + (X^1)^2 - (X^2)^2 - (X^3)^2 = \frac{1}{\Lambda}. \tag{3.7}$$

Based on the solution to the monodromies of the torus we have an algebraic equation more than the standard condition of embedding:

$$X_0{}^2 + X_1{}^2 - X_2{}^2 - X_3{}^2 = \frac{1}{\sqrt{\Lambda}}, \tag{3.8}$$

which is given by

$$\frac{X_0^2 - X_3^2}{cos^2(\sqrt{\Lambda}t)} = \frac{X_1^2 - X_2^2}{sin^2(\sqrt{\Lambda}t)}. \tag{3.9}$$

This condition can be interpreted as orthogonality condition between X^A and V^A :

$$\begin{aligned} X^A &= (X^0, X^1, X^2, X^3) \\ V^A &= \sqrt{\Lambda}\left[-tg(\sqrt{\Lambda}t)X^0, cotg(\sqrt{\Lambda}t)X^1, cotg(\sqrt{\Lambda}t)X^2, -tg(\sqrt{\Lambda}t)X^3\right]. \end{aligned} \tag{3.10}$$

If we apply many times from a generic point (X^0, X^1, X^2, X^3) the transformations (U, V) connected to the cycles (a, b) of the torus, we obtain a lattice of points which belong to the surface:

$$X_0^2 - X_3^2 = \frac{\sqrt{1-C^2}}{\sqrt{\Lambda}} \qquad X_1^2 - X_2^2 = \frac{C}{\sqrt{\Lambda}}. \tag{3.11}$$

At a given C such a surface describes again a torus, and the space-time evolution is simply obtained by allowing a time-dependent constant $C = C(t)$, which in the York time gauge is

$$C(t) = sin(\sqrt{\Lambda}t). \tag{3.12}$$

After a transformation of the flat coordinates

$$
\begin{aligned}
X^0 &= \frac{cos(\sqrt{\Lambda}t)}{\sqrt{\Lambda}}\cosh\left(\frac{\sqrt{\Lambda}Y}{cos(\sqrt{\Lambda}t)}\right) \qquad X^1 = \frac{sin(\sqrt{\Lambda}t)}{\sqrt{\Lambda}}\cosh\left(\frac{\sqrt{\Lambda}U}{sin(\sqrt{\Lambda}t)}\right)\\
X^2 &= \frac{sin(\sqrt{\Lambda}t)}{\sqrt{\Lambda}}\sinh\left(\frac{\sqrt{\Lambda}U}{sin(\sqrt{\Lambda}t)}\right) \qquad X^3 = \frac{cos(\sqrt{\Lambda}t)}{\sqrt{\Lambda}}\sinh\left(\frac{\sqrt{\Lambda}Y}{cos(\sqrt{\Lambda}t)}\right), \qquad (3.13)
\end{aligned}
$$

the lattice of points on the hyperboloid (3.8) becomes a plane lattice in the (U, Y) coordinates , which is analogous to a static torus. The rescaling t-dependent factor in front of U and Y is necessary to keep the spatial metric in conformal gauge:

$$
\begin{aligned}
ds^2 &= \left[1 - \Lambda\left(\frac{Y^2 sin^2(\sqrt{\Lambda}t)}{cos^2(\sqrt{\Lambda}t)} + \frac{U^2 cos^2(\sqrt{\Lambda}t)}{sin^2(\sqrt{\Lambda}t)}\right)\right]dt^2 + 2\frac{\sqrt{\Lambda}cos(\sqrt{\Lambda}t)}{sin(\sqrt{\Lambda}t)}UdUdt -\\
&- 2\frac{\sqrt{\Lambda}sin(\sqrt{\Lambda}t)}{cos(\sqrt{\Lambda}t)}YdYdt - dU^2 - dY^2, \qquad (3.14)
\end{aligned}
$$

from which we read the torus solution in the second order formalism $\alpha = 1$, $\beta = \frac{\sqrt{\Lambda}cos(\sqrt{\Lambda}t)}{sin(\sqrt{\Lambda}t)}U - i\frac{\sqrt{\Lambda}sin(\sqrt{\Lambda}t)}{cos(\sqrt{\Lambda}t)}Y$ and $e^{2\phi} = 1$.

The set of holonomy transformations (3.6) become the following ones which are pure translation monodromies ($\widetilde{Z} = U + iY$) :

$$
\begin{aligned}
\widetilde{Z} &\xrightarrow{a} \widetilde{Z} + \frac{1}{\sqrt{\Lambda}}[\lambda_1 sin(\sqrt{\Lambda}t) + i\lambda_2 cos(\sqrt{\Lambda}t)]\\
&\xrightarrow{b} \widetilde{Z} + \frac{1}{\sqrt{\Lambda}}[\eta_1 sin(\sqrt{\Lambda}t) + i\eta_2 cos(\sqrt{\Lambda}t)]. \qquad (3.15)
\end{aligned}
$$

The solution to them can be represented as a standard abelian integral on a z-plane with two branch cuts

$$
\widetilde{Z} = \int_0^z \frac{dz}{w(z,t)} \qquad w^2(z,t) = R(t)z(z-1)(z-\xi(t)), \qquad (3.16)
$$

where the position of the third singularity $\xi(t)$ is time-dependent, in order to allow that the translation monodromies are time-dependent (3.15).

Therefore, the solution for the torus is given by the composition of the mapping (3.13) with the abelian integral mapping (3.16).

The holonomies (3.6) tell us that the motion of the branch points is apparently almost free in the X^A coordinates, as they move as geodesics of the hyperboloid (3.7). Let us recall that a generic geodesic motion of it is parametrized by the following equation:

$$\begin{aligned} X^A &= \frac{1}{\sqrt{\Lambda}}(c_0^A cos(\sqrt{\Lambda}t) + c_1^A sin(\sqrt{\Lambda}t)) \\ c_0^A c_{A0} &= 1 \qquad c_1^A c_{A1} = m \qquad c_0^A c_{A1} = 0 \end{aligned} \tag{3.17}$$

where m is equal to $1, 0, -1$ respectively for a timelike, null or spacelike geodesic.

Taking for example the particle in 0 at rest, the resulting values for the constants $c_i^A, i = 1, 2$ are the following:

$$\begin{aligned} c_0^A(0) &= (1,0,0,0) \qquad c_1^A(0) = (0,1,0,0) \\ c_0^A(1) &= (cosh(\frac{\lambda_2}{2},0,0,sinh(\frac{\lambda_2}{2})) \qquad c_1^A(1) = (0,cosh(\frac{\lambda_1}{2},sinh(\frac{\lambda_1}{2}),0) \\ c_0^A(\xi) &= (cosh(\frac{\eta_2}{2},0,0,sinh(\frac{\eta_2}{2})) \qquad c_1^A(1) = (0,cosh(\frac{\eta_1}{2},sinh(\frac{\eta_1}{2}),0). \end{aligned} \tag{3.18}$$

In the z-coordinates, only the motion of the third singularity $\xi(t)$ is necessary up to a suitable rescalings of the z-coordinate, while the other two can remain at rest.

It is straightforward to derive the equations of motion for the modulus of the torus and for the area, which take the usual form [7],[12]:

$$\tau(t) = \frac{\lambda_1 sin(\sqrt{\Lambda}t) + i\lambda_2 cos(\sqrt{\Lambda}t)}{\eta_1 sin(\sqrt{\Lambda}t) + i\eta_2 cos(\sqrt{\Lambda}t)} \qquad A(t) = \int dz d\overline{z}\ e^{2\phi} = |\frac{sin(2\sqrt{\Lambda}t)}{2}\ (\lambda_1\eta_2 - \lambda_2\eta_1)|. \tag{3.19}$$

The motion of the modulus is essentially a consequence of the free motion in flat coordinates of the branch points. It has also to satisfy another consistency condition [12], namely that the motion of the moduli must be geodesic with respect to the natural metric of the moduli space, the Weil-Petersson metric, which in the case of the torus is equivalent to the Poincaré metric of the upper half τ-plane:

$$ds_\tau^2 = \frac{d\tau d\overline{\tau}}{(Im\tau)^2}. \tag{3.20}$$

The form (3.19) of the solution for the modulus is consistent since it describes a circular arc in the moduli space, which is a geodesic of the Poincaré metric.

The corresponding dreibein $E_\mu^A = \partial_\mu X^A$ is given by (A = (0,1, x, y)) :

$$E_z^A = \frac{\sqrt{\Lambda}}{2}\left(\frac{X^3}{icos(\sqrt{\Lambda}t)}, \frac{X^2}{sin(\sqrt{\Lambda}t)}, \frac{X^1}{sin(\sqrt{\Lambda}t)}, \frac{X^0}{icos(\sqrt{\Lambda}t)}\right). \tag{3.21}$$

The conformal gauge condition $\left(E_z^A\right)^2 = 0$ is verified due to Eq. (3.9).

Let us rewrite the torus mapping in terms of the general parametrization

$$\begin{aligned} \begin{pmatrix} X^t & X^z \\ \overline{X}^z & \overline{X}^t \end{pmatrix} &= \frac{1}{\sqrt{\Lambda}} \sqrt[4]{\frac{\partial_z g \partial_{\overline{z}} \overline{f}}{\partial_z f \partial_{\overline{z}} \overline{g}}} \frac{e^{-iT}}{\sqrt{(1-f\overline{f})(1-g\overline{g})}} \begin{pmatrix} -f\overline{g} & if \\ i\overline{g} & 1 \end{pmatrix} + \\ &+ \frac{1}{\sqrt{\Lambda}} \sqrt[4]{\frac{\partial_z f \partial_{\overline{z}} \overline{g}}{\partial_z g \partial_{\overline{z}} \overline{f}}} \frac{e^{iT}}{\sqrt{(1-f\overline{f})(1-g\overline{g})}} \begin{pmatrix} 1 & -ig \\ -i\overline{f} & -\overline{f}g \end{pmatrix}. \end{aligned} \tag{3.22}$$

It is enough to choose

$$\begin{aligned} f &= th\left(\frac{\sqrt{\Lambda}U}{2sin(\sqrt{\Lambda}t)} + \frac{\sqrt{\Lambda}Y}{2cos(\sqrt{\Lambda}t)} \right) \\ g &= th\left(\frac{\sqrt{\Lambda}U}{2sin(\sqrt{\Lambda}t)} - \frac{\sqrt{\Lambda}Y}{2cos(\sqrt{\Lambda}t)} \right), \end{aligned} \tag{3.23}$$

from which

$$N(z) = \frac{\sqrt{\Lambda}}{2sin(2\sqrt{\Lambda}t)w^2} \tag{3.24}$$

and the whole solution reduces to a couple of first integrals of two $O(2,1)$-σ models:

$$\begin{aligned} \frac{\partial_z f \partial_z \overline{f}}{\left(1-f\overline{f}\right)^2} &= \frac{\sqrt{\Lambda}}{2} \frac{e^{-2i\sqrt{\Lambda}t}}{sin(2\sqrt{\Lambda}t)} N(z) \\ \frac{\partial_z g \partial_z \overline{g}}{\left(1-g\overline{g}\right)^2} &= \frac{\sqrt{\Lambda}}{2} \frac{e^{2i\sqrt{\Lambda}t}}{sin(2\sqrt{\Lambda}t)} N(z). \end{aligned} \tag{3.25}$$

So we have found that in the torus case both f and g are real and N is related to the quadratic holomorphic differential. In general, we can guess that $N(z,t)$ is a known source for Eq. (3.25), being a combination of the quadratic holomorphic differentials of the Riemann surface.

The function tanh can also be expected since it diagonalizes the monodromy conditions for f around the cycles $C_i = (a,b)$ of the torus:

$$f \xrightarrow{C_i} \frac{A_i f + B_i}{B_i f + A_i} \qquad A_i = \cosh\frac{\chi_i}{2} \quad B_i = \sinh\frac{\chi_i}{2} \qquad i = 1,2. \tag{3.26}$$

The linear dependence of its argument from the abelian integrals represents the change of sign $f \to -f$ around each branch point.

Since f and g are real in the case of the torus and their image are contained inside the unit disk, they map a real variable inside the real diameter $D = [-1, 1]$. Since f and g are polydrome, they can be restricted to cover a segment of D. Circling many times around each cycle of the torus, $Image\ f$ and $Image\ g$ give a tessellation of the diameter.

This particular feature of the torus should be valid in general. For every Riemann surface the image of the maps f and g can be restricted to a polygon inside the corresponding unit disk. Circling many times around each cycle of the Riemann surfaces we should obtain a tessellation of the two unit disks, instead of their diameters, as a consequence of the nonabelian relation (3.5) which replaces the abelian one $UV = VU$ for the torus.

Another interesting case is the Λ-mapping for static Riemann surfaces, which is the case $f = g, T = 2\sqrt{\Lambda}t$, where the Λ-mapping reduces to :

$$\begin{aligned} X^A &= \frac{1}{\sqrt{\Lambda}}\left[cos(2\sqrt{\Lambda}t), sin(2\sqrt{\Lambda}t)n^i\right] \\ V^A &= \left[-sin(2\sqrt{\Lambda}t), cos(2\sqrt{\Lambda}t)n^i\right], \\ n^i &= \left(\frac{1+f\overline{f}}{1-f\overline{f}}, \frac{2\overline{f}}{1-f\overline{f}}, \frac{2f}{1-f\overline{f}}\right). \end{aligned} \tag{3.27}$$

A similar mapping has been discussed in [19]. The particular relation (3.27) between flat X^A-coordinates and f realizes explicitly the isomorphism between the subgroup $SO(2,1)$ of $SO(2,2)$ and $SU(1,1)$, and the Möbius transformations of f correspond to the $SO(2,1)$ holonomies for the X^A coordinates.

We easily recognize that the flat coordinates satisfy the constraint:

$$(X^1)^2 - X^z X^{\overline{z}} = \frac{sin^2(2\sqrt{\Lambda}t)}{\Lambda}. \tag{3.28}$$

The same surface can be obtain starting from a generic point X_0^A and applying all the $SO(2,1)$ movements related to the cycles (a_i, b_i). Hence this time foliation is natural since it is induced by the holonomies.

The $X = X(x)$ mapping (3.27) produces the hyperbolic metric on the disk:

$$ds^2 = 4dt^2 - 4\frac{sin^2 T}{\Lambda}\frac{df d\overline{f}}{(1-|f|^2)^2}, \tag{3.29}$$

from which we can read $\alpha = 2$, $\beta = 0$ and $e^{2\phi} = 4\frac{sin^2 T}{\Lambda}/(1-|f|^2)^2$ which solve eq. (2.8).

The conformal mapping $f(\overline{z})$ still has to be determined. Firstly, we remember that every genus g Riemann surfaces is determined by a fundamental group $\pi_1(\Sigma)$ generated by the $2g$ holonomies (U_i, V_i) satisfying the relation (3.5). Let us denote with H the unit disk with its metric of negative constant curvature. The group $SU(1,1)$ acts on it, maintaining its metric. Consider a subgroup of it $\Gamma \subset SU(1,1)$, isomorphic to $\pi_1(\Sigma)$, the quotient H/Γ is a Riemann surface of genus g, with the same constant curvature metric of the unit disk H.

Therefore, we can think that the image of the conformal mapping f gives a tessellation of the unit disk on which the holonomy acts as $SU(1,1)$ and we can restrict the fundamental region of Imf to be inside a closed geodesic $4g$-gon of the unit disk with hyperbolic metric, where the conformal mapping becomes one to one.

For pure $SO(2,1)$ holonomies, the equations of motion for the branch points are simply trivial timelike geodesic of the constraint $X^A X_A = \frac{1}{\Lambda}$.

Since the $X^A = X^A(x)$ mapping in Eq. (3.27) has also a trivial time dependence, we conclude that there is no evolution in the z-coordinate for the branch points. As a consequence, there is no time evolution for the moduli [12], and the only time dependence comes from the overall scale factor $sin^2 T$ in Eq. (3.29). We can suppose that the branch points ξ_i have fixed positions on the real axis. Then to find the Gauss map f it is helpful this theorem of complex analysis: given a simple and closed geodesic polygon Π of the Poincaré metric in the upper half w-plane, whose sides are circular arcs making angles $\alpha_1\pi$, $\alpha_2\pi$, ..., $\alpha_n\pi$, at the vertices $A_1, A_2, ..., A_n$ where $0 \leq \alpha_j \leq 2$, then there exist real numbers $\xi_1, \xi_2, ...\xi_n$, $\beta_1, \beta_2, ...\beta_n$ such that

$$\xi_1 < \xi_2 < ... < \xi_n, \qquad \sum_{j=1}^{n} \beta_j = 0$$
$$\sum_{j=1}^{n}(2\beta_j\xi_j + 1 - \alpha_j^2) = 0, \qquad \sum_{j=1}^{n}(\beta_j\xi_j^2 + (1-\alpha_j^2)\xi_j) = 0 \tag{3.30}$$

and the upper z-plane is conformally mapped inside Π by

$$w(z) = \frac{u_1(z)}{u_2(z)}, \tag{3.31}$$

where $u_1(z)$ and $u_2(z)$ are two linearly independent solutions of the Fuchsian differential equation:

$$u'' + \left[\frac{1}{4}\sum_{j=1}^{n}\frac{1-\alpha_j^2}{(z-\xi_j)^2} + \frac{1}{2}\sum_{j=1}^{n}\frac{\beta_j}{z-\xi_j}\right] u = 0. \tag{3.32}$$

The points ξ_j correspond to the vertices A_j. A simple conformal mapping relates the upper w-plane endowed with the Poincaré metric to the f-unit disk with the hyperbolic metric.

Let us remark that in order to obtain the tessellation property of the unit disk, the angles $\alpha_i\pi$ must be chosen as $\pi(1-1/n_i)$, with n_i positive integers. Such a mapping problem has been investigated in connection to Fuchsian groups and functions. In fact, to each geodesic polygon of the f-unit disk, whose angles in the vertices have the measure $\pi(1-1/n_i)$, with n_i positive integers, it is connected a discrete group of movements Γ and an analytic function $z(f)$ defined in the f-unit disk, which is invariant under the Γ action on the f variable.

$$z\left(\frac{a_k f + b_k}{\bar{b}_k f + \bar{a}_k}\right) = z(f) \qquad \Longleftrightarrow \qquad z(X^A) = z(\Lambda_B^{(k)A} X^B) \quad \forall k. \tag{3.33}$$

The Γ group of SU(1,1) Möbius transformations on the disk is called Fuchsian group. The function $z(f)$ is a Fuchsian function with respect to the group Γ. Instead the inverse $f = f(z)$ is polydrome, and it can be restricted to map the z-plane into the geodesic polygon. Therefore we conclude that our conformal mapping is the inverse of such fuchsian function $z(f)$. Examples of them can be built from the Poincaré series, having a simple covariant transformation under the Fuchsian group, from which we can obtain a Fuchsian function, invariant under the action of the Fuchsian group.

4 General solution for the immersion equation

The example of the torus can teach us a lot of information for the general case. We are going to discuss that we have already found the more general equations that constrain the dynamics of every Riemann surface.

Let us introduce a set of notations to be more comprehensible. In general the $O(2,2)$ cuts can be decomposed as products of $SU(1,1)$ cuts as

$$\begin{pmatrix} X^t & X^z \\ \overline{X}^z & \overline{X}^t \end{pmatrix} \to \begin{pmatrix} A_1 & B_1 \\ \overline{B}_1 & \overline{A}_1 \end{pmatrix} \begin{pmatrix} X^t & X^z \\ \overline{X}^z & \overline{X}^t \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ \overline{B}_2 & \overline{A}_2 \end{pmatrix}. \tag{4.1}$$

To make the monodromies more explicit we solved in [21] the constraint $X^A X_A = \frac{1}{\Lambda}$ with a parametrization that carries esplicit projective representation of each $SU(1,1)$:

$$f \to \frac{A_1 f + B_1}{\overline{B}_1 f + \overline{A}_1} \qquad g \to \frac{A_2 g + B_2}{\overline{B}_2 g + \overline{A}_2}. \tag{4.2}$$

It turns out that the following choice is relative simple and general:

$$
\begin{aligned}
X^A &= \frac{1}{\sqrt{\Lambda}}(h\tilde{W}^A + h^{-1}W^A) = \frac{1}{\sqrt{\Lambda}}(\overline{h^{-1}}\tilde{U}^A + \overline{h}U^A) \qquad h = e^{\frac{\phi_g - \phi_f}{2} - iT} \\
V_0^A &= i(-h\tilde{W}^A + h^{-1}W^A) = i(-\overline{h^{-1}}\tilde{U}^A + \overline{h}U^A)
\end{aligned}
\tag{4.3}
$$

where we define the following vectorial functions of f and g, which have not real components:

$$
\begin{aligned}
W^A &= \sqrt{\frac{\partial_{\overline{z}} g}{\partial_z f}}\frac{(1, -\overline{g}f, -i\overline{g}, -if)}{1 - g\overline{g}} \qquad \tilde{W}^A = \sqrt{\frac{\partial_{\overline{z}} f}{\partial_z g}}\frac{(-\overline{f}g, 1, i\overline{f}, ig)}{1 - f\overline{f}} \\
U^A &= \sqrt{\frac{\partial_z \overline{f}}{\partial_z \overline{g}}}\frac{(1, -\overline{g}f, -i\overline{g}, -if)}{1 - f\overline{f}} \qquad \tilde{U}^A = \sqrt{\frac{\partial_z \overline{g}}{\partial_z \overline{f}}}\frac{(-\overline{f}g, 1, i\overline{f}, ig)}{1 - g\overline{g}},
\end{aligned}
\tag{4.4}
$$

but the vectors X^A and V_0^A have real components, being coordinates. This global vector covariance can be useful in characterizing the scalar product of these vectors and their derivatives in terms of the invariants under the global SO(2,2) monodromy group.

As an example we can list the following identities :

$$
\begin{aligned}
W \cdot \tilde{W} &= U \cdot \tilde{U} = 1/2 \\
\tilde{W} \cdot \partial_z W &= \frac{1}{2}(\overline{H}_g - \overline{H}_f) \qquad h^{-1}\partial_z h = \frac{1}{2}\partial_z(\phi_g - \phi_f) - i\partial_z T \\
\tilde{W} \cdot U &= (W \cdot \tilde{U})^{-1} = \frac{e^{\phi_f - \phi_g}}{2}.
\end{aligned}
\tag{4.5}
$$

We also introduce the definition of the various invariants that will be useful in the following :

$$
\begin{aligned}
I_f &= \frac{\partial_z f \partial_z \overline{f}}{1 - f\overline{f}} \\
e^{2\phi_f} &= \frac{\partial_z \overline{f} \partial_{\overline{z}} f}{1 - f\overline{f}} \qquad e^{2\Omega_f} = \frac{\partial_z f \partial_{\overline{z}} \overline{f}}{1 - f\overline{f}} \\
H_f &= \frac{1}{2}\frac{\partial_z \partial_{\overline{z}} f}{\partial_z f} + \frac{\overline{f}\partial_{\overline{z}} f}{1 - f\overline{f}} \qquad \overline{H}_f = \frac{1}{2}\frac{\partial_z \partial_{\overline{z}} f}{\partial_{\overline{z}} f} + \frac{\overline{f}\partial_z f}{1 - f\overline{f}} \\
H_{\overline{f}} &= \overline{\overline{H}_f} \quad \overline{H}_{\overline{f}} = \overline{H_f},
\end{aligned}
\tag{4.6}
$$

and analogously for g.

Now let ask ourself the following question: can V_0^A be identified with the vector V^a which appears in the immersion equations (3.4) ? The answer is negative and positive at the same time, negative if we decide to work with general unconstrained fields, positive in practice, since if we look for a simple gauge fixed solution then it is possible to make such identification. Let us work out the difference:

$$V^A = \rho(V_0^A + \overline{\gamma}\partial_z X^A + \gamma\partial_{\overline{z}} X^A) \tag{4.7}$$

in which the coefficients γ and ρ are derived in such a way to satisfy the properties of V^A:

$$\begin{aligned}
\gamma &= 2e^{-2\phi}V_0^A \cdot \partial_z X^A = \frac{2i}{\sqrt{\Lambda}}e^{-2\phi}[\partial_z ln(h) - \overline{H}_g + \overline{H}_f] \\
\rho^2 &= \frac{1}{1+\gamma\overline{\gamma}e^{2\phi}}.
\end{aligned} \tag{4.8}$$

Moreover, it will useful in the following to compute the derivative of this vector $\partial_z V_0^A$, that can be again parametrized in terms of the basis of the 4 four-vectors

$$\partial_z V_0^A = \gamma_0 X^A + \gamma_1 V^A + \gamma_2 \partial_z X^A + \gamma_3 \partial_{\overline{z}} X^A \tag{4.9}$$

where

$$\begin{aligned}
\gamma_0 &= -\frac{\gamma}{2}e^{2\phi} \\
\gamma_1 &= -\rho^{-2}\partial_z\rho - \overline{\gamma}N(z) - \frac{k}{2}\gamma e^{2\phi} \\
\gamma_2 &= k\rho^{-1} - e^{-2\phi}\partial_z(\overline{\gamma}e^{2\phi}) \equiv \widetilde{k} \\
\gamma_3 &= 2\rho^{-1}e^{-2\phi}N(z) - \partial_z\gamma \equiv 2\widetilde{N}e^{-2\phi}.
\end{aligned} \tag{4.10}$$

The immersion equations (3.4) are covariant, and their information can be encoded in corresponding scalar products,

$$\begin{aligned}
(\partial_z X^A)^2 &= 0 \qquad \partial_z V^A \cdot \partial_z X^A = -N(z) \qquad (\partial_z V^A)^2 = -2kN(z) \\
\partial_z X^A \cdot \partial_{\overline{z}} X^A &= -\frac{1}{2}e^{2\phi} \qquad \partial_z V^A \cdot \partial_{\overline{z}} X^A = -\frac{k(t)}{2}e^{2\phi} \qquad \partial_z V^A \cdot \partial_{\overline{z}} V^A = -\frac{k(t)^2}{2}e^{2\phi} - 2N\overline{N}e^{-2\phi}.
\end{aligned} \tag{4.11}$$

Let us develop such requirements in two separate steps, the first one is using only the first line of eq. (4.11)

$$(\partial_z X^A)^2 = 0 \qquad \partial_z V_0^A \cdot \partial_z X^A = -\tilde{N}(z) \qquad (\partial_z V_0^A)^2 = -2\tilde{k}\tilde{N}(z) + \frac{\gamma_0^2}{\Lambda} + \gamma_1^2 \tag{4.12}$$

where we have decided to dress the basic fields $N(z)$ and the constant k with terms coming from γ:

$$\begin{aligned} \tilde{N} &= \rho^{-1}N(z) - \frac{1}{2}e^{2\phi}\partial_z\gamma \\ \tilde{k} &= k\rho^{-1} - e^{-2\phi}\partial_z(\overline{\gamma}e^{2\phi}). \end{aligned} \tag{4.13}$$

Inserting the general parametrization (4.3) in the first series of identities (4.12) we get the following constraints:

$$\begin{aligned} \partial_z W \cdot \partial_z \tilde{W} &= \frac{1}{2}\left[(\partial_z ln(h))^2 - 2\partial_z ln(h)(\overline{H}_g - \overline{H}_f) - \widetilde{N}(z)\widetilde{k}\right] + \frac{1}{4\Lambda}\gamma_0^2 + \frac{\gamma_1^2}{4} \\ (\partial_z W)^2 &= h^2\left[\frac{\widetilde{k} + i\sqrt{\Lambda}}{2}\widetilde{N}(z) - \frac{1}{4\Lambda}\gamma_0^2 - \frac{\gamma_1^2}{4}\right] \\ (\partial_z \widetilde{W})^2 &= h^{-2}\left[\frac{\widetilde{k} - i\sqrt{\Lambda}}{2}\widetilde{N}(z) - \frac{1}{4\Lambda}\gamma_0^2 - \frac{\gamma_1^2}{4}\right]. \end{aligned} \tag{4.14}$$

On the other hand the explicit computation of these scalar product based on their definition gives the following identities

$$\begin{aligned} \partial_z W \cdot \partial_z \tilde{W} &= -\frac{1}{2}(\overline{H}_f - \overline{H}_g)^2 - \frac{1}{2}(I_f + I_g) \\ (\partial_z \tilde{W})^2 &= \frac{\partial_z g}{\partial_{\overline{z}} g}e^{2\phi_f} = h^{-4}e^{-4iT}I_g \\ (\partial_z W)^2 &= \frac{\partial_z f}{\partial_{\overline{z}} f}e^{2\phi_g} = h^4 e^{4iT}I_f. \end{aligned} \tag{4.15}$$

Putting all the information together we finally obtain the following fundamental relations:

$$\begin{aligned} (\partial_z ln(h) + \overline{H}_f - \overline{H}_g)^2 &= I_f(h^2 e^{4iT} - 1) + I_g(h^{-2}e^{-4iT} - 1) \\ \widetilde{N} &= \frac{h^2 e^{4iT}I_f - h^{-2}e^{-4iT}I_g}{i\sqrt{\Lambda}} \\ \frac{\gamma_0^2}{4\Lambda} + \frac{\gamma_1^2}{4} &= \frac{\widetilde{k} - i\sqrt{\Lambda}}{2i\sqrt{\Lambda}}h^2 e^{4iT}I_f - \frac{\widetilde{k} + i\sqrt{\Lambda}}{2i\sqrt{\Lambda}}h^{-2}e^{-4iT}I_g. \end{aligned} \tag{4.16}$$

In the torus case these relations are solved by

$$I_f = \frac{\sqrt{\Lambda}}{2}\frac{e^{-iT}}{sinT}N(z) \qquad I_g = \frac{\sqrt{\Lambda}}{2}\frac{e^{iT}}{sinT}N(z). \tag{4.17}$$

It seems not possible to think that these equations satisfy with only one more relation all the immersion identities, but in reality it is like this, and the purpose of computing the second series of identities is to show that eqs. (4.17) are completely self-consistent.

If one choose the case $f = g$ these equations reduce to the following ones

$$\begin{aligned}
(\partial_z T)^2 &= -I_f(e^{iT} - e^{-iT})^2 \\
\widetilde{k} &= -\sqrt{\Lambda}\left[\frac{1}{tg(2T)} + \frac{\frac{\gamma_0^2}{\Lambda} + \gamma_1^2}{4I_f sin(2T)}\right] \\
I_f &= \frac{\sqrt{\Lambda}\widetilde{N}}{2sin(2T)}.
\end{aligned} \tag{4.18}$$

This case is a little misleading, since in the point of view of the present article it only corresponds to $N(z) = 0$ and $I_f = I_g = 0$, in the final simplified gauge choice. However one can recover from it another result, stated in reference [22], valid for a different gauge choice $k = 0$, where such simplification is not possible.

Let us compute the following scalar products

$$\begin{aligned}
i)A &= \partial_z W^A \cdot \partial_{\overline{z}} U^A \\
ii)B &= \partial_z \tilde{W}^A \cdot \partial_{\overline{z}} U^A \\
iii)C &= \partial_z W^A \cdot \partial_{\overline{z}} \tilde{U}^A \\
iv)D &= \partial_z \tilde{W}^A \cdot \partial_{\overline{z}} \tilde{U}^A
\end{aligned} \tag{4.19}$$

by firstly solving the following system

$$\begin{aligned}
e^{2iT}A &+ e^{-2iT}D + e^{\phi_g - \phi_f}B + e^{\phi_f - \phi_g}C + \partial_z ln(h)(H_{\overline{f}} - H_{\overline{g}}) + \\
&+ \partial_{\overline{z}} ln(k)(\overline{H}_g - \overline{H}_f) - \partial_z ln(h)\partial_{\overline{z}} ln(k) = -\frac{\Lambda}{2}e^{2\phi} \\
e^{2iT}A &- e^{-2iT}D + e^{\phi_g - \phi_f}B - e^{\phi_f - \phi_g}C = i\sqrt{\Lambda}\frac{\gamma_2}{2}e^{2\phi} \\
e^{2iT}A &- e^{-2iT}D - e^{\phi_g - \phi_f}B + e^{\phi_f - \phi_g}C = i\sqrt{\Lambda}\frac{\overline{\gamma}_2}{2}e^{2\phi} \\
-e^{2iT}A &- e^{-2iT}D + e^{\phi_g - \phi_f}B + e^{\phi_f - \phi_g}C + \partial_z ln(h)(H_{\overline{f}} - H_{\overline{g}}) +
\end{aligned}$$

$$+ \quad \partial_{\overline{z}} ln(k)(\overline{H}_g - \overline{H}_f) - \partial_z ln(h)\partial_{\overline{z}} ln(k) = \frac{1}{\Lambda}\gamma_0\overline{\gamma}_0 + \gamma_1\overline{\gamma}_1 - (\gamma_2\overline{\gamma}_2 + \gamma_3\overline{\gamma}_3)e^{2\phi} \tag{4.20}$$

coming from the immersion equations (3.4)

$$\begin{aligned}
\partial_z X^A \cdot \partial_{\overline{z}} X^A &= -\frac{1}{2}e^{2\phi} = \alpha_1 \qquad \partial_z V_0^A \cdot \partial_{\overline{z}} X^A = -\frac{\gamma_2}{2}e^{2\phi} = \beta_1 \\
\partial_{\overline{z}} V_0^A \cdot \partial_z X^A &= -\frac{\overline{\gamma}_2}{2}e^{2\phi} = \beta_2 \qquad \partial_z V_0^A \cdot \partial_{\overline{z}} V_0^A = \frac{1}{\Lambda}\gamma_0\overline{\gamma}_0 + \gamma_1\overline{\gamma}_1 - \frac{1}{2}(\gamma_2\overline{\gamma}_2 + \gamma_3\overline{\gamma}_3)e^{2\phi} = \alpha_2 .
\end{aligned} \tag{4.21}$$

It is not difficult to compute their explicit expression based on their definition (4.3)

$$\begin{aligned}
A &= \frac{1}{2}e^{\phi_f+\phi_g}\left[1 + \frac{\partial_z f \partial_{\overline{z}}\overline{g}}{\partial_{\overline{z}} f \partial_z \overline{g}}\right] \\
B &= \frac{1}{2}e^{\phi_f-\phi_g}\left[(H_{\overline{f}} - H_{\overline{g}})(\overline{H}_f - \overline{H}_g) - e^{2\phi_f} - e^{2\Omega_g}\right] \\
C &= \frac{1}{2}e^{\phi_g-\phi_f}\left[(H_{\overline{f}} - H_{\overline{g}})(\overline{H}_f - \overline{H}_g) - e^{2\Omega_f} - e^{2\phi_g}\right] \\
D &= \frac{1}{2}e^{\phi_f+\phi_g}\left[1 + \frac{\partial_{\overline{z}}\overline{f}\partial_z g}{\partial_z \overline{f}\partial_{\overline{z}} g}\right] .
\end{aligned} \tag{4.22}$$

By comparing these equations with the result coming from the immersion we obtain finally :

$$\begin{aligned}
(\partial_z lnh + \overline{H}_f - \overline{H}_g)(\partial_{\overline{z}} lnk + H_{\overline{g}} - H_{\overline{f}}) &= -e^{2\phi_f} - e^{2\Omega_g} - \frac{\Lambda\alpha_1 + \alpha_2}{2} + i\sqrt{\Lambda}\frac{\beta_1 - \beta_2}{2} = \\
&= -e^{2\phi_g} - e^{2\Omega_f} - \frac{\Lambda\alpha_1 + \alpha_2}{2} - i\sqrt{\Lambda}\frac{\beta_1 - \beta_2}{2} \\
e^{\phi_f+\phi_g}\left[1 + \frac{\partial_z f}{\partial_{\overline{z}} f}\frac{\partial_{\overline{z}}\overline{g}}{\partial_z \overline{g}}\right] &= \frac{e^{-2iT}}{2}\left[\Lambda\alpha_1 - \alpha_2 - i\sqrt{\Lambda}(\beta_1 + \beta_2)\right] \\
e^{\phi_f+\phi_g}\left[1 + \frac{\partial_{\overline{z}}\overline{f}}{\partial_z \overline{f}}\frac{\partial_z g}{\partial_{\overline{z}} g}\right] &= \frac{e^{2iT}}{2}\left[\Lambda\alpha_1 - \alpha_2 + i\sqrt{\Lambda}(\beta_1 + \beta_2)\right] .
\end{aligned} \tag{4.23}$$

Now let us discuss these identities in detail. For example if we make the assumption that $f = real$ and $g = real$ as in the torus case, we obtain the following identities $A = \overline{D}$ and $B = C$. In particular if we substitute directly the torus solution we get:

$$A = \frac{\Lambda cosT e^{-iT}}{4w\overline{w}sin^2T}$$

$$
\begin{aligned}
B &= C = -\frac{\Lambda}{4w\overline{w}sin^2T} \\
D &= \overline{A} = \frac{\Lambda cosTe^{iT}}{4w\overline{w}sin^2T}.
\end{aligned} \tag{4.24}
$$

In the general case $f = g$ by definition we get

$$
A = -B = -C = D, \tag{4.25}
$$

while on the other hand we obtain that

$$
\begin{aligned}
A &= \frac{e^{-2iT}}{4}\left[\Lambda\alpha_1 - \alpha_2 - i\sqrt{\Lambda}(\beta_1 + \beta_2)\right] \\
B &= \frac{e^{\phi_f - \phi_g}}{2}\left[\partial_z lnh \partial_{\overline{z}} lnk + \frac{\Lambda\alpha_1 + \alpha_2}{2} - i\sqrt{\Lambda}\frac{\beta_1 - \beta_2}{2}\right] \\
C &= \frac{1}{2}\left[\partial_z lnh \partial_{\overline{z}} lnk + \frac{\Lambda\alpha_1 + \alpha_2}{2} + i\sqrt{\Lambda}\frac{\beta_1 - \beta_2}{2}\right] \\
D &= \frac{e^{2iT}}{4}\left[\Lambda\alpha_1 - \alpha_2 + i\sqrt{\Lambda}(\beta_1 + \beta_2)\right].
\end{aligned} \tag{4.26}
$$

The compatibility of these two equations (still for $f = g$) implies that

$$
\begin{aligned}
Im(\gamma_2) &= 0 \\
\sqrt{\Lambda}\gamma_2 &= tg(2T)\left[\left(\frac{\gamma_2\overline{\gamma}_2 + \gamma_3\overline{\gamma}_3}{2} - \frac{\Lambda}{2}\right) - \left(\frac{\gamma_0\overline{\gamma}_0}{\Lambda} + \gamma_1\overline{\gamma}_1\right)e^{-2\phi})\right] \\
\partial_z ln(h)\partial_{\overline{z}} ln(h) &= -\frac{e^{2\phi}}{4}\sqrt{\Lambda}\left[\sqrt{\Lambda} - tgT\gamma_2\right].
\end{aligned} \tag{4.27}
$$

Finally the torus case and the static Riemann surface case have in common the following simple solution of all these constraints, namely that $T = T(t)$ and $\gamma = 0$, and by consequence that $\gamma_0 = \gamma_1 = 0$, $\gamma_2 = k$ e $\gamma_3 = 2e^{-2\phi}N(z)$, and moreover that $\widetilde{N} = N, \widetilde{k} = k$.

5 The gauge $\gamma = 0$

In $2 + 1$ gravity the gauge conditions $k = 0$ and $g_{zz} = 0$ were enough to fix completely the fields. Here the same remark cannot be applied. The reason for this strange behaviour of AdS gravity lies in the fact that are no natural boundary conditions on the fields at infinity, therefore we have more freedom to fix the gauge.

We are going to choose the boundary conditions in such a way that the equations of motion can be expressed by the simplest choice. It is not difficult to find such a gauge. In fact, we can be inspired by the simplest solutions that we have at disposition, the solutions for the torus and for the static Riemann surfaces, which have in common the auxiliary condition $\gamma = 0$. If we take seriously such a condition in general, for every Riemann surface, we obtain a drastic simplification in the expression for the equations of motion, practically the simplest one that we have could written obeying all the physical requirements we have discussed so far:

$$I_f e^{iT} = I_g e^{-iT} = \frac{\sqrt{\Lambda}N(z)}{2sinT} \qquad H_f = H_g = 0. \tag{5.1}$$

Moreover it is true that $\gamma_0 = \gamma_1 = 0$, $\widetilde{k} = k$, $\widetilde{N} = N$ and $T = 2\sqrt{\Lambda}t$, $\quad k = \frac{\sqrt{\Lambda}}{tg(T)}$.

We know that an analitic solution of the monodromy conditions for f and g is pratically impossible for particles, because it gives rise to many problems, for example the fact that the horizon defined by the condition $|f| = 1$ doesn't match in generally with the condition $|g| = 1$, which is a quite important constraint, to know where the spatial slice of the universe ends, or the problem how to give motion to the particles, since for an analitic solution the geodesic equations are undefined (see [21] for an analysis of such problems). For reasons of symmetry with the particle case we believe that analogous problems would also appear for Riemann surfaces.

This is the simplest equation for a non-analytic solution to the monodromies, analogous to what we have discussed for the torus, a couple of first integrals of two $O(2,1)$ sigma models.

Moreover both the first set of equations (4.12) coming from the embedding equations (3.4) that the second one (4.21) are solved if we add to the system of conditions (5.1) another fundamental equation:

$$e^{2\phi_f} = e^{2\phi_g} = \frac{\Lambda}{4sin^2T} e^{2\phi}. \tag{5.2}$$

In this way, the static equation of Sine-Gordon type for ϕ is solved automatically as in the case of the torus, and moreover the following integral condition holds:

$$\int_S \nabla^2\phi = 4\int_S \frac{df \wedge d\overline{f}}{1-|f|^2} = 4\int_S \frac{dg \wedge d\overline{g}}{1-|g|^2}. \tag{5.3}$$

6 A corollary: scattering of black holes

As a corollary for these set of simplifyed equations we can consider the scattering of back holes [20], in which the gauge choice can be taken again as the condition $\gamma = 0$ with the only change that $T \to T + \frac{\pi}{2}$.

One of the problems that were bothering me about building a solution for the scattering of black holes, was the fact the gauge choice conformal $g_{zz} = 0$ cannot be necessarily global but it can be extended only until the region external to the horizons, where the time is well defined and distinct from a spatial coordinate. Therefore there is not only the spatial end of the universe, but also two internal horizons on which the condition $|f| = |g| = 1$ must hold.

Necessarily we have to impose that the locus of points defined by the first condition $|f| = 1$ coincides with the locus of points defined by the second condition $|g| = 1$. But this requirement of physical consistency is already inside the equation of consistency of the gauge $\gamma = 0$:

$$\phi_f = \phi_g, \tag{6.1}$$

because whenever the first condition holds implies that $e^{2\phi_f}$ is divergent as $1/(1-|f|^2)$, and this equation (6.1) automatically requires that there is an analogous divergency of $e^{2\phi_g}$ as $1/(1-|g|^2)$. Therefore the set of equations proposed (5.1) and (5.2) is complete and must contain the physically consistent solution for the scattering of black holes. There are no extra conditions to be added since these are already contained in the equation (6.1).

We can suggest how to solve the monodromy conditions directly, by requiring that our unknown f is analytic with respect to an intermediate variable w (this partial analyticity tell us how to makes sense to the word cut in a non-analytic setting),

$$f = f(w(z,\overline{z},t), \qquad f(w) = w(z,\overline{z},t)^{c-1} \frac{F(a,b,c;w(z,\overline{z},t))}{F(a-c+1,b-c+1,2-c:w(z,\overline{z},t))}, \tag{6.2}$$

which is single-valued with respect to the physical complex variable z, but it is also a non trivial reparametrization of it. A quite non trivial constraint on the choice for the reparametrization $w(z,\overline{z},t)$ comes from the requirement that the whole solution satisfies the $O(2,1)$ sigma model.

In this way the problem of defining the geodesic equations for the particles or the black holes is reduced to find the time-dependent part of this reparametrization, which at the end is responsible for their motion. Instead the monodromy conditions are already satisfied at all orders by a careful choice of the coefficients for the hypergeometric function. We will show in a forecoming paper what type of solution is obtained, by doing perturbation with respect to the parameter Λ [23].

7 Discussion

We have shown here that the first-order formalism, that has allowed us to solve the N-body problem, can be extended to the case of Riemann surfaces in $(2+1)$ Gravity with cosmological constant.

We have found how to recover from the solution of two $O(2,1)$ σ-model a general solution of all Einstein equations. The solution is also characterized by an analytic function $N(z)$ which is the component K_{zz} of the extrinsic curvature tensor. The two solutions of the $O(2,1)$ sigma model map the z-complex plane with branch cuts into a direct product of unit disks with hyperbolic metric. The holonomies of f and g are elements of $SU(1,1) \otimes SU(1,1)$ and isometries of the corresponging hyperbolic metric. Therefore we can delimitate $Imf \otimes Img$ into a direct product of geodesic polygons inside the two unit disk. This property is not generally true for the analogous mappings $f$and g of the N-particle, since in that case there is a boundary to the spatial slice of the universe, ad the limit $|f| = 1$ and $|g| = 1$ are reached. In the case of the scattering of black holes these limit are reached not only to represent the spatial infinity but also their horizons. Instead the peculiar signal of topology should be that these mappings produce a tessellation of the two unit disks, a property which is not true for the particle case. The line defined by $|f| = 1$ is not reached by the Riemann surface solutions.

We have given explicit solutions for the mapping functions f and g for the torus and for all Riemann surfaces having $SO(2,1)$ holonomies. It turns out that in both cases the inverse mapping $z = z(f)$ is a single-valued function, i.e. an automorphic function. The $N(z)$ function for the torus is related to the quadratic holomorphic differential, a property that is probably true for all Riemann surfaces.

We have found that the $SO(2,2)$ holonomies have a quite simple particle interpretation. They determine the evolution for the branch points, which move as timelike geodesic of the (3.8) hyperboloid in the embedding flat four-dimensional space-time.

The moduli trajectories, which have to be geodesic of the metric of Teichmuller space, should be a consequence of the geodesic trajectories of the branch points.

The next step would be to find a quantization scheme which takes into account this classical reduction of three dimensional gravity in two-dimensional field theories.

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A Appendix - First-order formalism in the limit $\Lambda \to 0$

We would like to clarify how the second-order equations are solved by the simplified choice of the first-order solution. Let us notice firstly that the conformal factor can be written in two equivalent ways:

$$
\begin{aligned}
e^{2\phi} &= \frac{N\overline{N}}{\partial_z f \partial_{\overline{z}} \overline{f}} (1 - f\overline{f})^2 = \frac{N\overline{N}}{\partial_z g \partial_{\overline{z}} \overline{g}} (1 - g\overline{g})^2 = \\
&= \frac{4sin^2 T}{\Lambda} \frac{\partial_z \overline{f} \partial_{\overline{z}} f}{(1 - f\overline{f})^2} = \frac{4sin^2 T}{\Lambda} \frac{\partial_z \overline{g} \partial_{\overline{z}} g}{(1 - g\overline{g})^2} \\
I_f &= \frac{\partial_z f \partial_z \overline{f}}{(1 - f\overline{f})^2} = \frac{\sqrt{\Lambda} e^{-iT}}{2sinT} N(z) \qquad I_g = \frac{\partial_z g \partial_z \overline{g}}{(1 - g\overline{g})^2} = \frac{\sqrt{\Lambda} e^{iT}}{2sinT} N(z).
\end{aligned} \tag{A.1}
$$

The equation for the conformal factor ϕ is solved because it can be written as:

$$
\nabla^2 \phi = 4 \frac{\partial_z f \partial_{\overline{z}} \overline{f} - \partial_z \overline{f} \partial_{\overline{z}} f}{(1 - f\overline{f})^2} = 4 \frac{\partial_z g \partial_{\overline{z}} \overline{g} - \partial_z \overline{g} \partial_{\overline{z}} g}{(1 - g\overline{g})^2}. \tag{A.2}
$$

In this way it is pratically identical to the equation of motion that we solved for the torus case in $(2+1)$gravity (see Appendix of ref. [11]). The only difference of this case is that the same mechanism of solution is repeated twice, once for the function f and the other time for the function g.

Analogously the equation for $\alpha$is solved by looking at the definition of the field α in terms of first-order quantities:

$$
\partial_z \partial_{\overline{z}} \alpha = \partial_z \partial_{\overline{z}} V^A \cdot \partial_0 X^a + \partial_z V^A \cdot \partial_0 \partial_{\overline{z}} X^A + \partial_{\overline{z}} V^A \cdot \partial_0 \partial_z X^A + V^A \cdot \partial_z \partial_{\overline{z}} X^A. \tag{A.3}
$$

By rearranging the embedding equations (3.4) we can find the following properties:

$$
\begin{aligned}
\partial_z \partial_{\overline{z}} V^a &= \left(2N\overline{N} e^{-2\phi} + \frac{k^2}{2} e^{2\phi} \right) V^A + \frac{\Lambda}{2} k(t) e^{2\phi} X^A \\
V^a \cdot \partial_0 \partial_z \partial_{\overline{z}} X^A &= \frac{\Lambda}{2} e^{2\phi} \alpha + \partial_0 \left(\frac{k(t)}{2} e^{2\phi} \right) \\
\partial_z V^A \cdot \partial_0 \partial_{\overline{z}} X^A &= k(t) \partial_z X^A \cdot \partial_0 \partial_{\overline{z}} X^A.
\end{aligned} \tag{A.4}
$$

By summing up all the contribution the second order equation for α is naturally solved.

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